Water waves of finite amplitude on a sloping beach

By G. F. CARRIER and H. P. GREENSPAN
Pierce Hall, Harvard University

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SUMMARY

In this paper, we investigate the behaviour of a wave as it climbs a sloping beach. Explicit solutions of the equations of the non-linear inviscid shallow-water theory are obtained for several physically interesting wave-forms. In particular it is shown that waves can climb a sloping beach without breaking. Formulae for the motions of the instantaneous shoreline as well as the time histories of specific wave-forms are presented.

1. INTRODUCTION

The behaviour of waves on sloping beaches has received intensive study by many authors during the past sixty years. These investigations, for the most part, have been confined to studies of linearized problems which are based on assumptions that are invalid in the neighbourhood of the coastline. With the results of these linear theories as a basis, it has been stated that progressing waves eventually break on a sloping beach.

In this paper, we present an analysis based on the non-linear shallow-water theory. Explicit solutions are obtained for a number of important cases and, in particular, it is shown that there are waves that climb a sloping beach without breaking. The initial shape and particle velocity distribution determine whether or not a given wave will break, and no simple criterion for the occurrence of breaking has been found.

2. GENERAL ANALYSIS

The conservation equations of mass and momentum of the non-linear shallow-water theory are

$$[v^*(\eta^*+h^*)]_{x^*} = -\eta^*_{t^*}, \tag{2.1}$$

$$v^*_{t^*}+v^*v^*_{x^*} = -g\eta^*_{x^*}, \tag{2.2}$$

where v^* is the horizontal velocity and the other symbols are defined in figure 1; the asterisks denote dimensional quantities. A complete development of the non-linear shallow-water theory can be found in Stoker (1948).

It is convenient to introduce the following dimensionless quantities: $v = v^*/v_0$, $\eta = \eta^*/\alpha l_0$, $x = x^*/l_0$, $t = t^*/T$, $c^2 = (h^*+\eta^*)/\alpha l_0$. In these definitions, $T = (l_0/\alpha g)$ and $v_0 = (gl_0\alpha)^{1/2}$. The characteristic length l_0 can be specified when a specific problem is adopted for study; the depth is assumed to be of uniform slope, $h^* = -\alpha x^*$.

With these substitutions, equations (2.1) and (2.2) become

$$v_t + vv_x + \eta_x = 0, \tag{2.3}$$

$$[v(\eta - x)]_x + \eta_t = 0. \tag{2.4}$$

These hyperbolic equations can be rewritten in a form in which the characteristic variables α, β play the role of the independent variables and v, c, x and t play the role of the unknown functions of α, β. The four equations which arise when the classical transformation is made (the details are given in Stoker (1948)) are

$$x_\beta - (v+c)t_\beta = 0, \qquad x_\alpha - (v-c)t_\alpha = 0, \qquad (2.5),\ (2.6)$$

$$v_\beta + 2c_\beta + t_\beta = 0, \qquad v_\alpha - 2c_\alpha + t_\alpha = 0. \qquad (2.7),\ (2.8)$$

Equations (2.7) and (2.8) can be integrated explicitly to obtain

$$v + 2c + t = \alpha, \qquad v - 2c + t = -\beta. \qquad (2.9),\ (2.10)$$

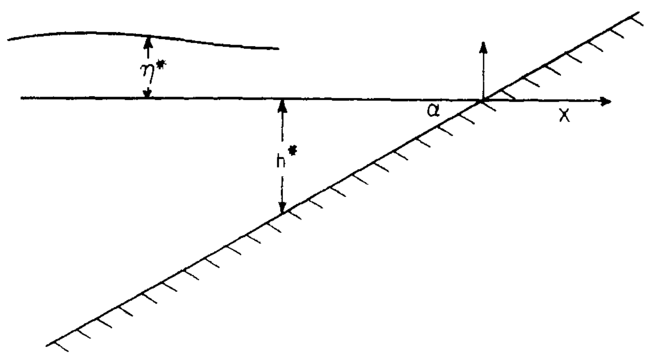

Figure 1. Definition sketch. The fluid has a sloping fixed boundary and a free surface at height η^* above its undisturbed level.

Here, the 'constants of integration' have been chosen in the interest of algebraic simplicity in what follows. From (2.9) and (2.10) we obtain

$$v + t = (\alpha - \beta)/2 = \lambda/2, \tag{2.11}$$

$$c = (\alpha + \beta)/4 = \sigma/4; \tag{2.12}$$

and these define λ and σ. We now adopt λ and σ as our final pair of independent variables, so that (2.5) and (2.6) become

$$x_\sigma - vt_\sigma + ct_\lambda = 0, \tag{2.13}$$

$$x_\lambda + ct_\sigma - vt_\lambda = 0. \tag{2.14}$$

The elimination of x results in the *linear* second-order equation for t

$$\sigma(t_{\lambda\lambda} - t_{\sigma\sigma}) - 3t_\sigma = 0; \tag{2.15}$$

and, since $v + t = \lambda/2$, v must also satisfy (2.15). In fact, it is readily verified by reference to equations (2.11) to (2.14) that if we introduce the 'potential' $\phi(\sigma, \lambda)$ so that

$$v = \sigma^{-1}\phi_\sigma(\sigma, \lambda), \tag{2.16}$$

then

$$x = \phi_\lambda/4 - \sigma^2/16 - v^2/2, \tag{2.17}$$

$$\eta = c^2 + x = \phi_\lambda/4 - v^2/2, \tag{2.18}$$

$$t = \lambda/2 - v, \tag{2.19}$$

and

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0. \tag{2.20}$$

Two major simplifications have been obtained. The non-linear set of equations (2.5) through (2.8) have been reduced to a linear equation for v or ϕ and the free boundary (the instantaneous shoreline $c = 0$, which moves as a wave climbs a beach) is now the fixed line $\sigma = 0$ in the (σ, λ)-plane.

The choice of a function $\phi(\sigma, \lambda)$ which satisfies equation (2.20) defines η, v, x, t in terms of the parametric coordinates σ, λ. In particular, if the Jacobian $\partial(x,t)/\partial(\sigma,\lambda)$ never vanishes in $\sigma > 0$, the implicitly defined solutions $\eta(x,t)$ and $v(x,t)$ are single-valued, and such solutions represent waves which do not break. If the foregoing Jacobian does vanish in $\sigma > 0$, the wave must break. However, we confine our attention in this paper to those forms of ϕ for which breaking does not occur.

A particularly simple solution of these equations is given by

$$\phi = AJ_0(\omega\sigma)\cos(\omega\lambda - \psi), \tag{2.21}$$

where J_0 is the usual notation for a Bessel function. No loss in generality ensues when the phase lag ψ is taken to be zero or when ω is put equal to unity, so that

$$\phi = AJ_0(\sigma)\cos\lambda.$$

The Jacobian $J = \partial(x,t)/\partial(\sigma,\lambda)$ vanishes nowhere in $\sigma > 0$ when $A \leqslant 1$ and the mapping is valid in $\sigma \geqslant 0$. The physical phenomenon whose description is implied by equation (2.21) is that which occurs when a wave of unit frequency in the dimensionless time variable travels shoreward from the region of large x and is reflected, so that a wave, again of unit frequency, travels out to sea. The reflection coefficient is unity. The phenomenon is periodic in the time variable and the wave shape far at sea is like $J_0(4\sqrt{|x|})$, but it is considerably distorted near the shore. In particular, the penetration of the wave (the value of x at which the depth is zero) is given by equation (2.17) when $\sigma = 0$: i.e. $x(\lambda, 0) = \phi_\lambda/4 - u^2/2$, so the maximum penetration is $A/4$. When $A > 1$, J vanishes on some curve in $\sigma > 0$, and the solution must be modified so that a bore is included in the prediction. The analysis of that problem is now being studied, but will not be discussed here. The wave shape is shown in figures 2 and 3, for the extreme positions corresponding to $\lambda = \pi/2$ and $\lambda = 3\pi/2$ both for $A = 1$ and $A = \frac{1}{2}$. In the limit as $A \to 0$, $J_0(\sigma)$ becomes $J_0(4\sqrt{|x|})$, the linearized solution, and no graph should be needed.

We now consider problems in which a mound of water is released: that is, we specify a wave shape $\eta(x, 0)$ with $v(x, 0) = 0$ everywhere. Since $v + t = \lambda/2$, the condition that $v = 0$ when $t = 0$ implies that $\lambda = 0$ for $t = 0$. Equation (2.15) for v must be solved in the region $\sigma \geqslant 0$, $\lambda \geqslant 0$, with the initial conditions that $v = 0$ and v_λ is specified on $\lambda = 0$, and the boundary condition that v is to be finite on $\sigma = 0$.

The derivative v_λ can be determined from the prescribed initial wave height by first solving the equation

$$[\eta(x,0)-x]^{1/2} = c(x,0) = \tfrac{1}{4}\sigma$$

for x as a function of σ, and then using equation (2.13) to show that on $\lambda = 0$

$$x_\sigma = -ct_\lambda.$$

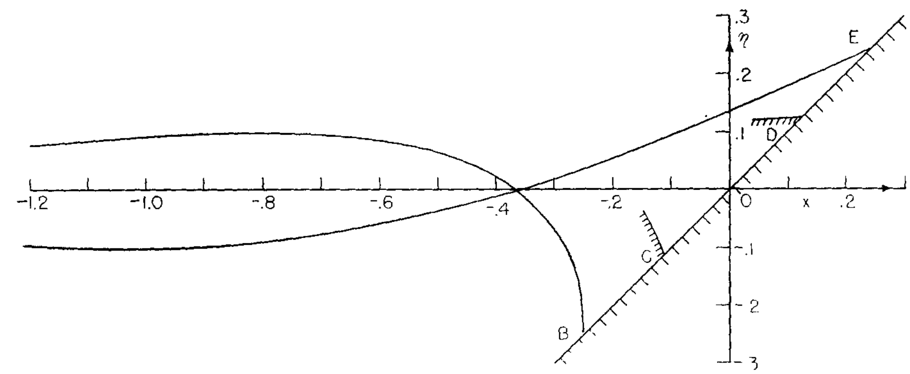

Figure 2. Free surface geometry for periodic motion according to equation (2·21) for $A = 1$ at: E, point of maximum penetration, $\lambda = \pi/2$; B, $\lambda = 3\pi/2$; C, D, intermediate times.

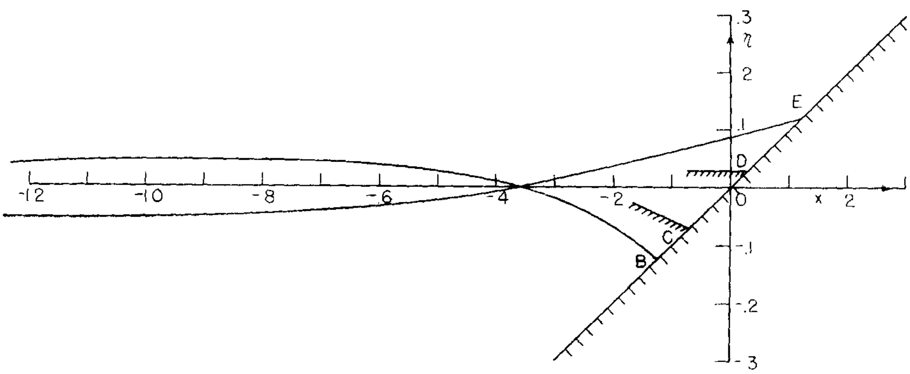

Figure 3. Free surface for periodic motion according to equation (2.21) for $A = \frac{1}{2}$ at: E, $\lambda = \pi/2$; B, $\lambda = 3\pi/2$; C, D, intermediate times.

Therefore, for $\lambda = 0$,

$$v_\lambda = \tfrac{1}{2} - t_\lambda = \tfrac{1}{2} + 4\sigma^{-1}x_\sigma = f(\sigma).$$

An entirely equivalent boundary-value problem can be formulated for the potential ϕ.

The solution of the boundary-value problem for the particle velocity v is easily obtained by transform techniques. Let

$$\bar{v} = \int_0^\infty e^{-s\lambda} v \, d\lambda;$$

then equation (2.15) becomes

$$\sigma \bar{v}_{\sigma\sigma} + 3\bar{v}_{\sigma} - s^2 \sigma \bar{v} = -\sigma f(\sigma). \tag{2.22}$$

Let $z = s\sigma$ and $p = z\bar{v}$; then

$$(zp')' - \frac{p}{z} - zp = -\frac{z^2}{s^2} f(z/s), \tag{2.23}$$

and if we now use a Hankel transform

$$\bar{p} = \int_0^\infty z J_1(\xi z) p(z)\, dz,$$

we obtain after further manipulations

$$\bar{p} = \frac{1}{1+\xi^2} \int_0^\infty \frac{\beta^2}{s^2} J_1(\beta s) f(\beta/s)\, d\beta.$$

Upon setting $\xi = s\tau$ and computing the inverse transforms, there results

$$v = \int_0^\infty \sigma^{-1} J_1(\tau\sigma) \sin \tau\lambda\, d\tau \int_0^\infty \sigma_0^2 J_1(\tau\sigma_0) f(\sigma_0)\, d\sigma_0, \tag{2.24}$$

or, in terms of the potential,

$$\phi = -\int_0^\infty \tau^{-1} J_0(\tau\sigma) \sin \tau\lambda\, d\tau \int_0^\infty \sigma_0^2 J_1(\tau\sigma_0) f(\sigma_0)\, d\sigma_0. \tag{2.25}$$

These results can also be obtained by the superposition of standing waves. The function $\sigma^{-1} J_1(k\sigma) \sin k\lambda$ is a solution of (2.21). By the principle of superposition,

$$v = \int_0^\infty A(k) \sigma^{-1} J_1(k\sigma) \sin k\lambda\, dk$$

is also a solution. The function $A(k)$ is determined from the boundary condition $v_\lambda = f(\sigma)$ on $\lambda = 0$; the condition that $v = 0$ on $\lambda = 0$ is implicitly satisfied. Further reduction of these general expressions leaving $f(\sigma)$ unspecified does not simplify the task of evaluating the final integrals. Instead, we select functions $f(\sigma)$ which will both simplify the final integrals and correspond to physically interesting initial wave shapes.

3. Some initial value problems

In this section we consider a number of interesting examples of wave propagation problems in which the motion starts from rest at time zero. As a first example, let a one-parameter family of wave-forms at $t = 0$ be given by

$$\eta = \epsilon \left[1 - \frac{5}{2} \frac{a^3}{(a^2+\sigma^2)^{3/2}} + \frac{3}{2} \frac{a^5}{(a^2+\sigma^2)^{5/2}} \right], \tag{3.1}$$

$$x = -\frac{\sigma^2}{16} + \epsilon \left[1 - \frac{5}{2} \frac{a^3}{(a^2+\sigma^2)^{3/2}} + \frac{3}{2} \frac{a^5}{(a^2+\sigma^2)^{5/2}} \right], \tag{3.2}$$

where $a = 1{\cdot}5(1+0{\cdot}9\epsilon)^{1/2}$. These waves, shown in figure 4 for two values of ϵ, all have maximum heights equal to ϵ, all have heights $0{\cdot}9\epsilon$ at $x = -1$, and all have zero slopes at the shoreline. These shapes correspond to the physical problem in which the water level at the coastline is depressed, the fluid held motionless and then released. The quantity $f(\sigma)$ is found to equal

$$30a^3\epsilon\left[\frac{1}{(a^2+\sigma^2)^{5/2}}-\frac{a^2}{(a^2+\sigma^2)^{7/2}}\right],$$

and for this function it can be shown that

$$\int_0^\infty \sigma_0^2 J_1(\tau\sigma_0) f(\sigma_0)\, d\sigma_0 = 2a^2\epsilon\tau e^{-a\tau}(4-a\tau). \tag{3.3}$$

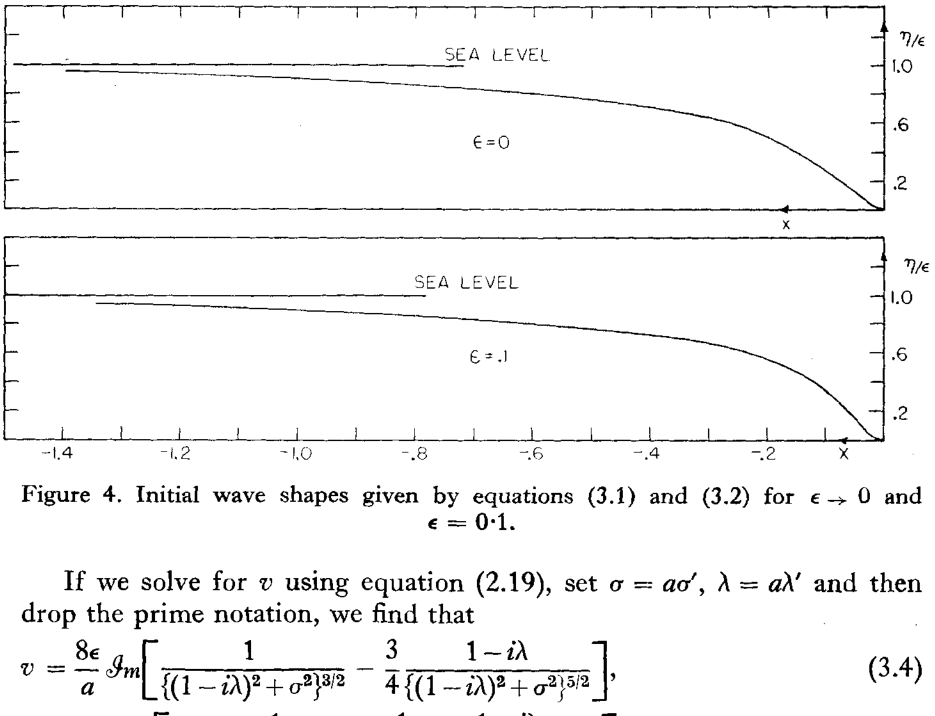

Figure 4. Initial wave shapes given by equations (3.1) and (3.2) for $\epsilon \to 0$ and $\epsilon = 0{\cdot}1$.

If we solve for v using equation (2.19), set $\sigma = a\sigma'$, $\lambda = a\lambda'$ and then drop the prime notation, we find that

$$v = \frac{8\epsilon}{a}\mathcal{I}m\left[\frac{1}{\{(1-i\lambda)^2+\sigma^2\}^{3/2}}-\frac{3}{4}\frac{1-i\lambda}{\{(1-i\lambda)^2+\sigma^2\}^{5/2}}\right], \tag{3.4}$$

$$\phi = 8\epsilon a\mathcal{I}m\left[\frac{-1}{\{(1-i\lambda)^2+\sigma^2\}^{1/2}}+\frac{1}{4}\frac{1-i\lambda}{\{(1-i\lambda)^2+\sigma^2\}^{3/2}}\right], \tag{3.5}$$

$$x = -\frac{v^2}{2}-\frac{a^2\sigma^2}{16}+\epsilon\mathcal{R}e\left[1-2\frac{5/4-i\lambda}{\{(1-i\lambda)^2+\sigma^2\}^{3/2}}+\frac{3}{2}\frac{(1-i\lambda)^2}{\{(1-i\lambda)^2+\sigma^2\}^{5/2}}\right], \tag{3.6}$$

$$t = \tfrac{1}{2}a\lambda - v, \qquad c = \tfrac{1}{4}a\sigma, \qquad (3.7),\ (3.8)$$

and

$$\eta = x + a^2\sigma^2/16. \tag{3.9}$$

The motion of the instantaneous shoreline, or zero depth position, is obtained by setting $\sigma = 0$, and is given by

$$v = \frac{8\epsilon}{a}\frac{5\lambda^3-\lambda^5}{(1+\lambda^2)^4}, \tag{3.10}$$

$$x = -\tfrac{1}{2}v^2+\epsilon-\frac{\epsilon}{(1+\lambda^2)^3}(1+3\lambda^2-2\lambda^4), \tag{3.11}$$

$$t = \tfrac{1}{2}a\lambda - v, \qquad c = 0. \qquad (3.12),\ (3.13)$$

The maximum penetration distance attained by the climbing wave occurs when the coastline velocity is zero, i.e. for $\lambda^2 = 5$, and is given by $x_{max} = 1{\cdot}157\epsilon$. That is, the water level, if depressed a depth $\epsilon\alpha l_0$ at the shoreline and then released, will rise to a height 15% higher than the original sea level. Figures 5, 6 and 7 present a time history of the action, and figure 8 is a plot of the position and velocity of the instantaneous

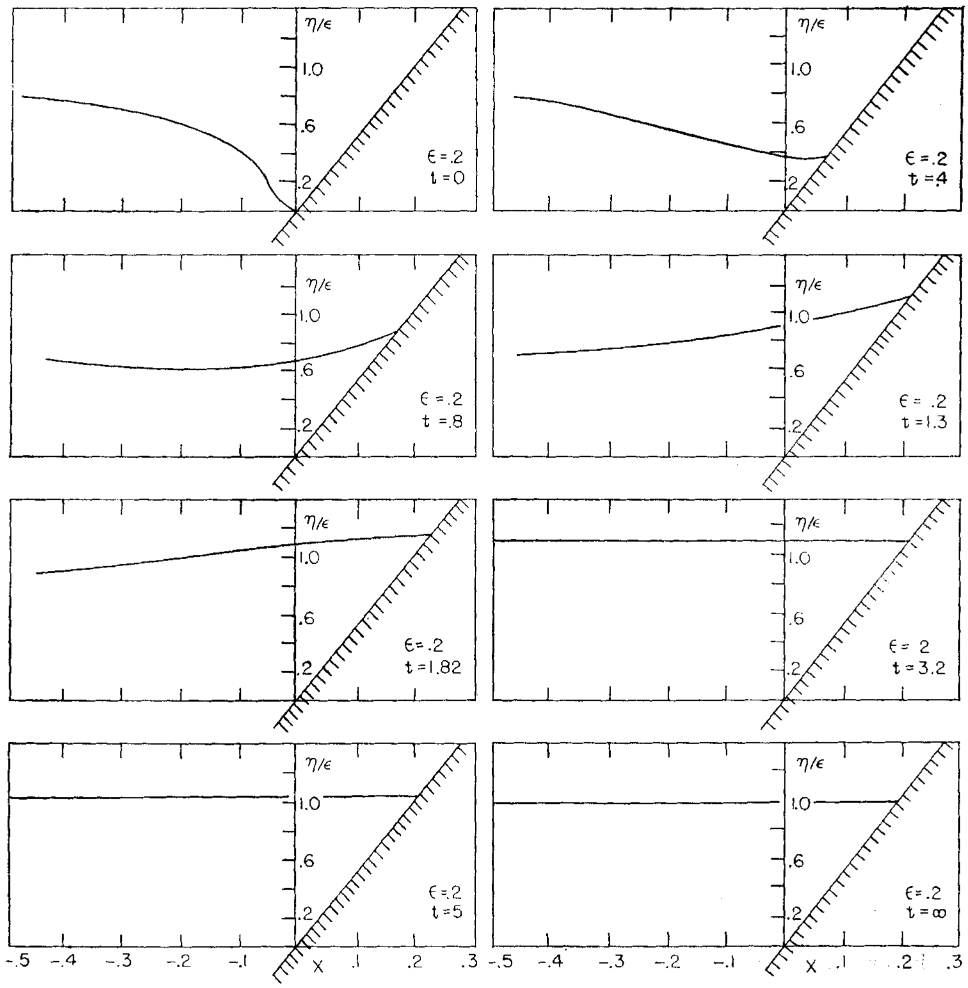

Figure 5. Time history of the wave-form of equation (3.1) for $\epsilon = 0{\cdot}2$, near the coastline.

shoreline for the specific wave shape $\epsilon = 0{\cdot}1$. It is seen that the instantaneous shoreline rises above the mean sea level and then slowly settles back to it. There are no continued oscillations about this position and the waves do not break provided ϵ is sufficiently small, namely $\epsilon \leqslant 0{\cdot}23$.

As a second example consider the motion of a stationary mound of water released at $t = 0$, which is given by

$$\eta = \tfrac{1}{4}\epsilon p^2 e^2 \sigma^4 e^{-\sigma^2 p}, \tag{3.14}$$

and

$$x = \tfrac{1}{4}\epsilon e^2 p^2 \sigma^4 e^{-\sigma^2 p} - \sigma^2/16, \tag{3.15}$$

where $1/p = 8(1+\epsilon)$. The family of wave-forms, shown in figure 9, all have their initial maximum wave heights at a fixed position from the shoreline,

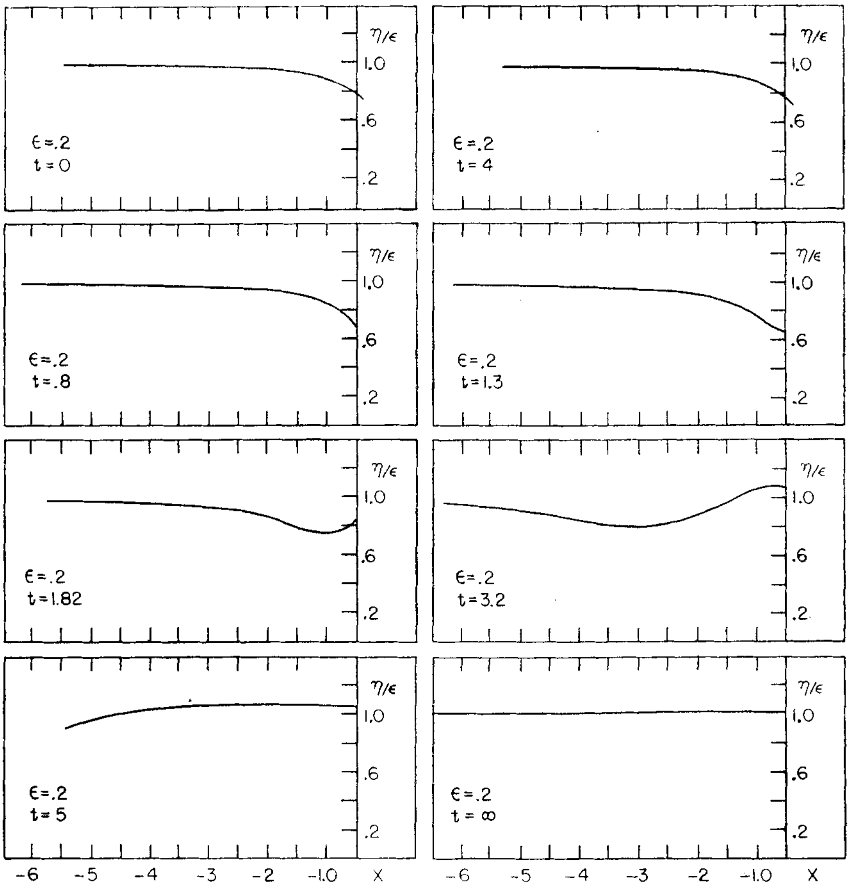

Figure 6. Time history of the wave-form of equation (3.1) for $\epsilon = 0{\cdot}2$, far from the coastline.

$\eta = \epsilon$ at $x = -1$. In addition all have zero slope at the origin. The quantity v_λ evaluated at $\lambda = 0$ is found to be

$$f(\sigma) = 2\epsilon e^2 p^2 (2\sigma^2 e^{-\sigma^2 p} - \sigma^4 p e^{-\sigma^2 p}); \tag{3.16}$$

and from Watson (1944, § 13.3) we find that

$$\int_0^\infty \sigma_0^{2n+2} J_1(\tau\sigma_0) e^{-\sigma_0^2 p}\, d\sigma_0 = (-1)^n \tau \frac{d^n}{dp^n}\left(\frac{1}{4p^2} e^{-\tau^2/4p}\right). \tag{3.17}$$

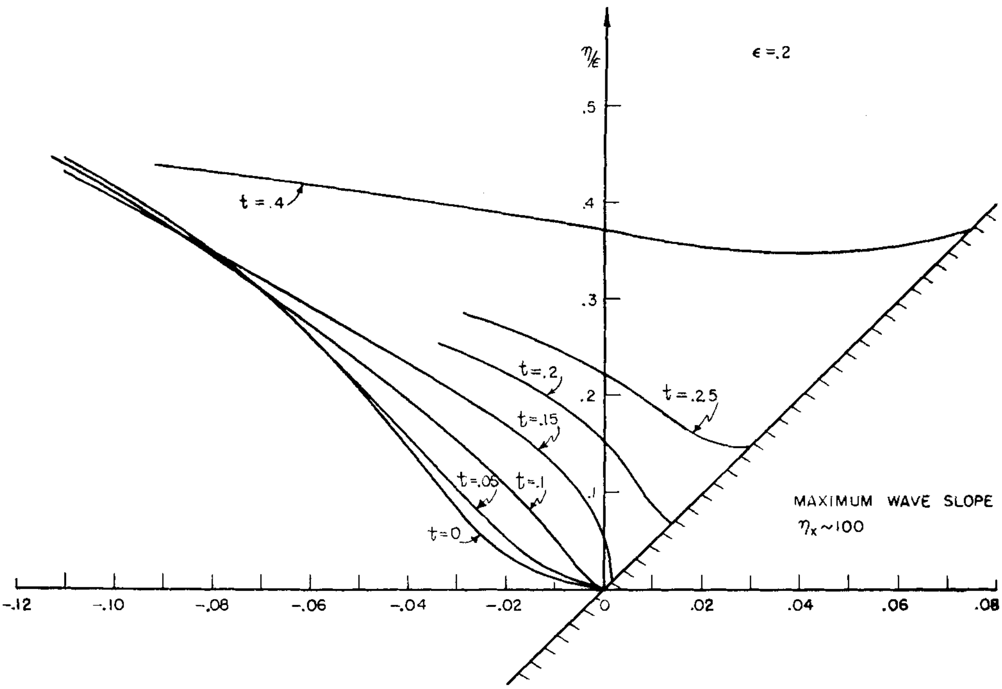

Figure 7. Time history of the wave-form of equation (3.1), $\epsilon = 0{\cdot}2$, in the neighbourhood of the beach.

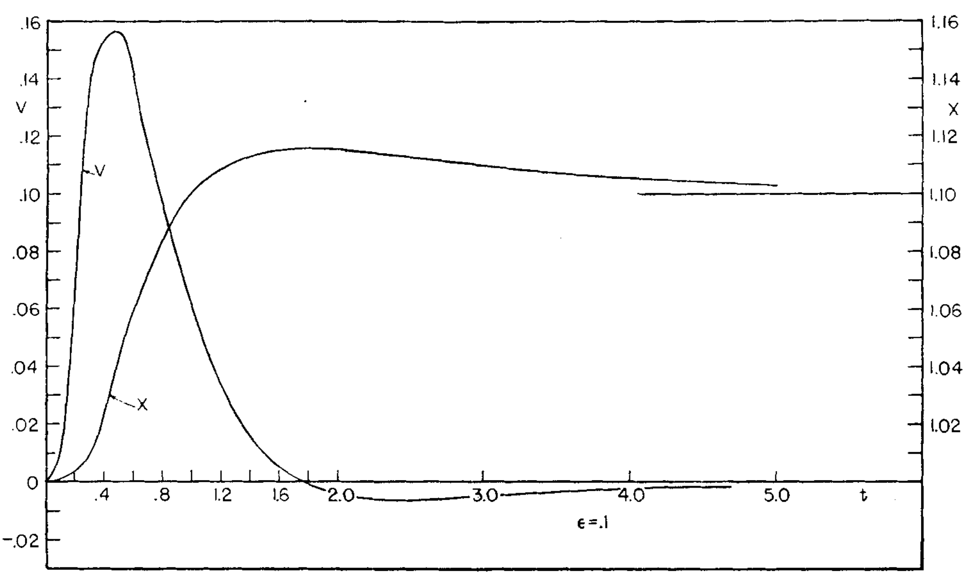

Figure 8. Coastline position and velocity *vs* time for the wave-form of equation (3.1) with $\epsilon = 0{\cdot}1$.

If we set $z = \frac{1}{2}\tau p^{-1/2}$, then the complete solution is

$$v = -4\epsilon e^2 \int_0^\infty z\sigma^{-1} J_1(2p^{1/2}\sigma z)\sin(2p^{1/2}\lambda z)e^{-z^2}(1-2z^2+\tfrac{1}{2}z^4)\,dz, \tag{3.18}$$

$$\phi = 2\epsilon e^2 p^{-1/2} \int_0^\infty \sin(2p^{1/2}\lambda z) J_0(2p^{1/2}\sigma z)e^{-z^2}(1-2z^2+\tfrac{1}{2}z^4)\,dz, \tag{3.19}$$

$$x = -v^2/2 - \sigma^2/16 + \epsilon e^2 \int_0^\infty z\cos(2p^{1/2}\lambda)e^{-z}\,(1-2z^2+\tfrac{1}{2}z^4)J_0(2p^{1/2}\sigma z)\,dz, \tag{3.20}$$

$$t = \lambda/2 - \phi_\sigma/\sigma, \quad \text{and} \quad c = \tfrac{1}{4}\sigma. \qquad (3.21),\ (3.22)$$

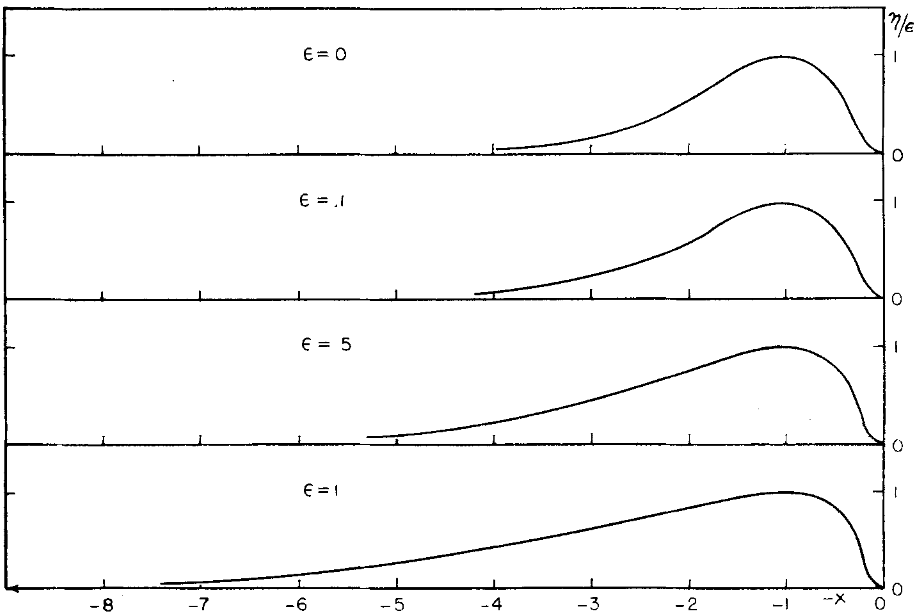

Figure 9. Exponential wave-forms of equation (3.15); $\epsilon = 0$, 0·1, 0·5, 1.

It is seen that the quantities v, v_λ and v_σ have upper bounds which are independent of σ and λ: i.e.

$$|v| < M\epsilon, \qquad |v_\lambda| < M_1\epsilon, \qquad |v_\sigma| < M_2\epsilon.$$

For ϵ sufficiently small, the Jacobian

$$|J| = \tfrac{1}{4}\sigma|v_\sigma^2 - (\tfrac{1}{2} - v_\lambda)^2|$$

is approximately given by $\sigma/16$, and therefore does not vanish in the interior of the fluid. In other words, for ϵ small enough, the wave forms of equations (3.14) and (3.15) do not break as they climb the beach. That the Jacobian is zero when σ is zero is a property of the transformations and is not a consequence of assuming a particular initial wave shape. The slope of the wave at the instantaneous shoreline, $\sigma = 0$, can remain finite even though the Jacobian vanishes. Similar statements hold for the wave shapes of the previous example.

If we restrict ourselves to a discussion of the motion of the shoreline, set $\lambda' = 2p^{1/2}\lambda$, $\sigma' = 2p^{1/2}\sigma$ and afterwards drop the prime notation, then the equations of motion of the shoreline can be written as

$$v = (\pi p)^{1/2}\epsilon e^2\, df(\lambda)/d\lambda, \tag{3.23}$$

$$t = \tfrac{1}{4}\lambda p^{-1/2} - (\pi p)^{1/2}\epsilon e^2\, df(\lambda)/d\lambda \tag{3.24}$$

$$c = 0, \qquad x = -v^2/16 + \epsilon e^2\pi^{1/2} f(\lambda)/4, \qquad (3.25),\ (3.26)$$

where

$$f(\lambda) = \frac{1}{16}\left[-\lambda^2 + \frac{\lambda^4}{2} + e^{-\lambda^2/4}E(\lambda)\left(\lambda + \lambda^3 - \frac{\lambda^5}{4}\right)\right], \tag{3.27}$$

$$\frac{df}{d\lambda} = \frac{1}{16}\left[-\lambda + 3\lambda^3 - \frac{\lambda^5}{4} + e^{-\lambda^2/4}E(\lambda)\left(1 + \frac{5\lambda^2}{2} - \frac{7\lambda^4}{4} + \frac{\lambda^6}{8}\right)\right], \tag{3.28}$$

and

$$E(\lambda) = \int_0^\lambda e^{y^2/4}\, dy. \tag{3.29}$$

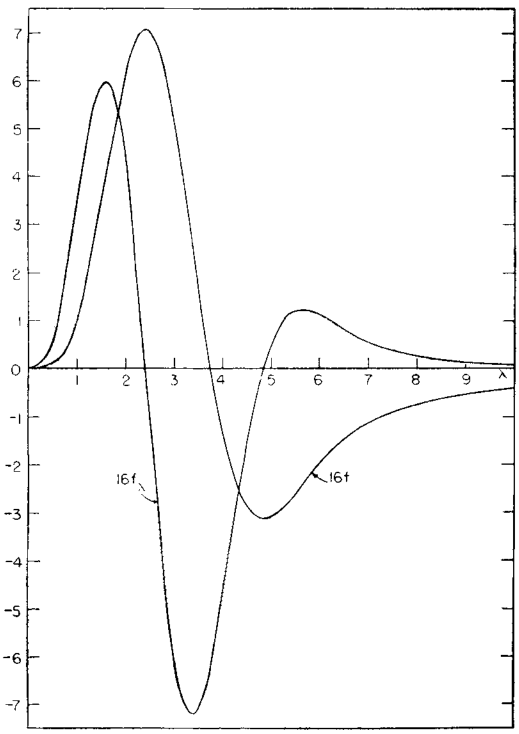

Figure 10. A plot of the functions $16f$ and $16f_\lambda$ *vs* λ.

The functions $16f$, $16f_\lambda$ are shown in figure 10; asymptotically $16f \sim -32/\lambda^2$, $16f_\lambda \sim 64/\lambda$, so that $v \sim 4(\pi p)^{1/2}\epsilon e^2/\lambda^3$. The coastline motion is such that it first rises to its maximum height, falls back below its initial position, and finally returns, very slowly, to the original mean sea level. There are no continued oscillations about the mean sea level. The maximum penetration distance occurs when $v = 0$, for $\lambda = 2{\cdot}41$ and is found to be $x_{max} = 1{\cdot}451\epsilon$. The maximum height attained is 45% greater than the maximum initial wave height. The coastline velocity is again zero at $\lambda = 4{\cdot}835$, which implies that the lowest depth reached is $x_{min} = -0{\cdot}636\epsilon$. The shoreline motion for the particular wave shapes $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}5$ are shown in figures 11 and 12.

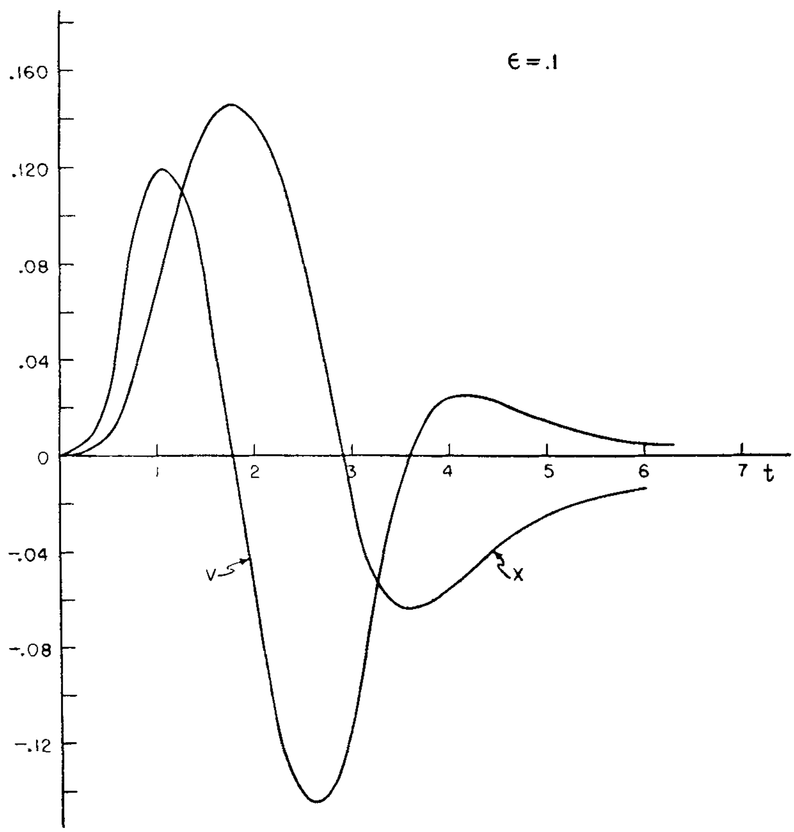

Figure 11. Coastline position and velocity *vs* time for the exponential wave $\epsilon = 0{\cdot}1$ of equation (3.15).

4. Conclusion

Thus far we have shown that there are progressing waves of a compressive nature (positive amplitude) which do not break as they climb a sloping beach. No general criteria have been found which enable us to determine when a given wave will break, although the magnitude of ϵ and hence the initial wave shape are of fundamental importance. The wave-forms we have considered all had continuous derivatives, the first derivative being zero, initially, at the coastline. Such waves do not break

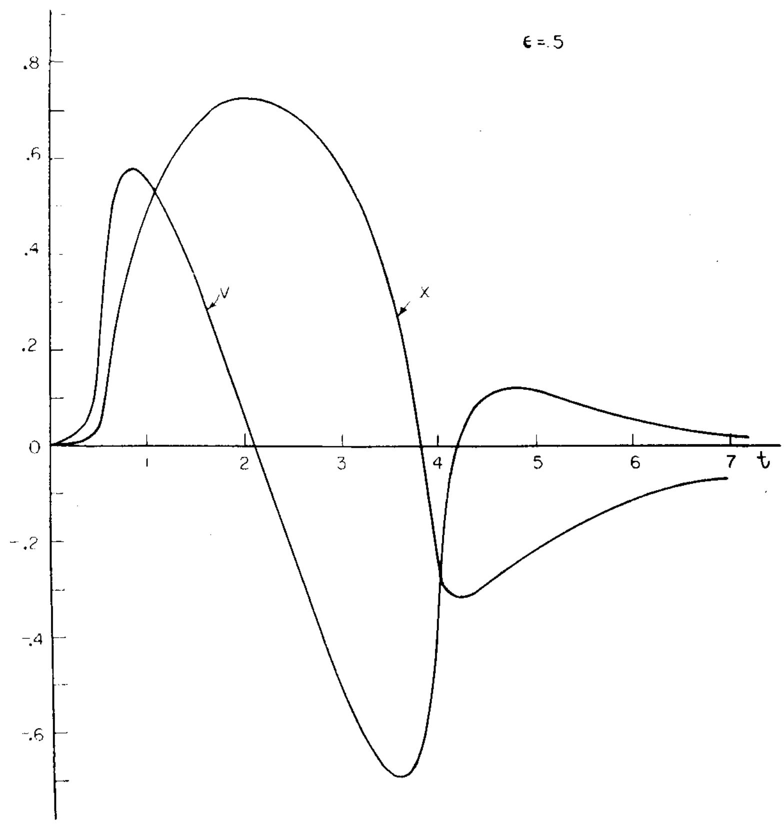

Figure 12. Coastline position and velocity *vs* time for the exponential wave $\epsilon = 0{\cdot}5$ of equation (3.15).

for sufficiently small ϵ. In a subsequent paper, it will be shown that all compressive waves (waves of positive amplitude) propagating into quiescent water towards the beach with a discontinuity in first derivative necessarily break before reaching the coastline.

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References

STOKER, J. J. 1948 The formation of breakers and bores, *Comm. Pure Appl. Math.* **1**, 1.

WATSON, G. N. 1944 *A Treatise on the Theory of Bessel Functions*, 2nd Ed. Cambridge University Press.